**Dot Product**

Dot product (or scalar product) can be described as the multiplication of two vectors in a certain direction. However, dot product goes beyond just multiplying two vectors together, as we will see in a moment.

To begin, when given two vectors =<a1, a2, a3> and =<b1, b2, b3>, the dot product of the two vectors can be calculated by the following equation:

• = a1b1+ a2b2+ a3b3

Example:

Compute the dot product for the following vectors:   
 =<4, 7, -3> and =<5, -6, 2>   
  
Using the equation for the dot product, we find that

• = (4\*2) + (7\*6) + (-3\*2)   
 = 8 + 42 + (-6)   
 = 48

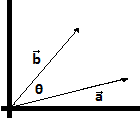
Similar to vector arithmetic, dot product has properties that can be proved computationally. It is encouraged that you attempt to prove some of the properties.

Properties of the Dot Product

Given three vectors , and along with the scalar c,

1. •( +)= • + •
2. • = •
3. • = IIII2
4. (c)• = •(c)= c ( •)
5. • = 0
6. If • =0 then =

Geometrically, the dot product can be represented by the following image. Suppose that θ is the angle between the vectors and such that 0 < θ < π.

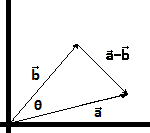


From this image, we can derive the following theorem.

Theorem:

• = IIII IIII cosθ

How do we derive this equation? First, let’s look again at the diagram, with an added vector:



Recall that Cosine Law states that given a triangle with sides of length a, b, c and angle acb= θ we have c2= a2 + b2 – 2abcosθ. Using this law, we can apply it to the triangle in the diagram. Note that the length of the sides are simply the magnitudes of the vectors.

From the Cosine Law, we have the following:

(1) II -II2 = IIII2 + IIII2- 2 IIII IIII cosθ  
Using the properties of dot product (specifically #3), we can re-write the left side of the equation:

II -II2 = ( -) • ( -)  
Using FOIL distribution, the brackets are expanded  
II -II2 = • - •- •+ •  
Using properties 2 and 3 we can simply this equation along with collecting like terms  
(2) II -II2 = IIII2- 2 •+ IIII2  
  
Now, equation (2) can be applied to equation (1)   
II -II2 = IIII2 + IIII2- 2 IIII IIII cosθ  
IIII2- 2 •+ IIII2= IIII2 + IIII2- 2 IIII IIII cosθ  
-2• = IIII2 + IIII2- 2IIII IIII cosθ - IIII2- IIII2  
-2• = - 2 IIII IIII cosθ   
• = IIII IIII cosθ

Notice that this equation can be applied to vectors in any dimension, so long the two vectors used are in the same dimension. Generally, the equation will be used to calculate the angle between two vectors, and not the dot product.

Example:   
Given the vectors = <2, 6, -3> and = < 5, -2, 1>, calculate the angle between the vectors.   
  
Knowing that the equation • = IIII IIII cosθ will be used, the magnitudes of the vectors along with the dot product are calculated.

• =2\*5+ 6\*(-3)+ (-3)\*1  
• = 10- 18- 3  
• =-11  
IIII=√ 22+ 62+ (-3)2  
IIII= √4+ 36+ 9  
IIII= 7  
IIII=√52+ (-2)2+ 12  
IIII=√25+ 4+ 1  
IIII=√30

Now we can make the appropriate substitutions.   
• = IIII IIII cosθ  
-11= 7\*√30 cosθ  
cosθ= -11/ (7\*√30)   
cosθ= -0.287  
θ= 106.67⁰

What are some applications of the dot product? Dot product can be used to determine whether two vectors are parallel or perpendicular to each other without the need to draw them on a plane. Moving forward, if two vectors are perpendicular to each other, they will be called orthogonal instead of perpendicular.

If two vectors are orthogonal, then the angle between them is 90⁰. Thus cos(90)=0 and • =0  
If two vectors are parallel, then then the angle between them is either 0⁰ or 180⁰. Therefore, if two vectors are parallel, then:

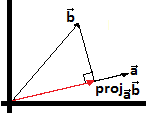
• = IIII IIII cos(0)  
• = IIII IIII (1)   
• = IIII IIII  
• = IIII IIII cos(180)   
• = IIII IIII (-1)   
• = -IIII IIII

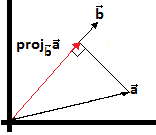
Example: Determine if the vectors = <1, -2> and = <1/4, -1/2> are parallel, orthogonal or neither.   
First we check if the two vectors are orthogonal by calculating the dot product  
• = 1(1/4) + (-1/2) (-2)   
• = ¼ +1  
• = 5/4 Since • doesn’t equal zero, the vectors are not orthogonal.   
  
Now we calculate IIII and IIII to determine if the vectors are parallel.

IIII=√ 12+ (-2)2  
IIII=√ 1+ 4  
IIII=√5  
III=√ (1/4)2+ (-1/2)2  
III=√ 1/16+ 1/4  
III=√ 5/16 = (√5)/4

Notice that IIII IIII= (√5) [(√5)/4] = 5/4. Therefore, • = IIII IIII (cosθ= 0) and the vectors are parallel.

A much quicker solution is to note that vector is the vector multiplied by a scalar value, in this case ¼. Therefore it can be said that if two vectors and are parallel, there exists a scalar k such that = k.   
  
Where is the dot product applied in terms of space travel and research? As previously mentioned, the orbit of a spacecraft is due to the addition of a velocity vector and the downward vector due to the force of gravity. However, what effect does the downward vector have on the velocity vector? Dot product is used to calculate this, and is generally thought of as ‘multiplication, with direction’.

Another application of dot product is in determining the projection of one vector onto another. When given two vectors and , the projection of onto is denoted by proj. Visually, the projection of onto is drawn in the following steps:   
1. Draw a line straight down from the head of vector until a right angle is formed with vector   
2. Draw a vector that is parallel to, starting at the same point as both original vectors and ending where the first line meets the vector .   
  


Note, that the projection of vectoronto would result in a completely different vector. Thus, the projection of  *onto*  is denoted as proj  


Visually  
Arithmetically, the projection of onto can be calculated using the formula:

proj= •   
 IIII2

Now it is important that the notations are not mixed, as the projection of onto can be calculated using another the formula:

proj= •   
 IIII2

Example: Determine the projection of vector*=* <5, 0, -7> onto= <2, -4, 1>   
As the question states the projection of onto , we will use the second equation.   
proj= •   
 IIII2

•= 5(2)+ 0(-4)+ (-7)(1)  
•= 10+0-7  
•= 3  
IIII2= (√22+(-4)2+ 12)2  
IIII2= (√4+16+ 1)2IIII2= (√21)2= 21

proj= <2, -4, 1>   
 21  
proj=<2/7, -4/7, 1/7>  
  
As additional practice, calculate the projection of onto .   
  
Problems

Calculate the dot product between the following vectors:   
1. <2, 3, 20> and <-13, -2, 6> = 88  
2. <-17, -10, 1> and <-6, -18, 7> = 275  
3. <-3, 9, 11> and <-9, -4, 19> = 200  
4. <-16, -5, -1> and <13, 6, 17> = -305  
5. <11, 15, 14> and <-7, -12, 10> = -37  
  
Determine the angle between the following vectors:   
1. <-14, 8, 12> and <-19, 0, 4>   
θ= 36.43⁰  
2. <-15, -20, 5> and <-8, -6, 12>  
θ= 41.12⁰  
3. <8, -8, 3> and <-15, -16, 2>  
θ=86.89⁰  
  
Are the following pairs of vectors parallel, orthogonal, or neither?   
1. <3, 4, 5> and <9/2, 6, 15/2> - Parallel  
2. <2, 5, 2> and < 6, -2, -1> - Orthogonal   
3. < -3, -9, -5> and <17, 16, -12> - Neither  
  
If <1, 2, k> and <2, 1, -1> are orthogonal, what is the value of k?  
If the vectors are orthogonal, then <1, 2, k> • <2, 1, -1>= 0  
1(2)+ 2(1)+ k(-1)= 0  
2+ 2- k= 0  
4- k=0   
k= 4  
  
If = 2- 3 and = 8+ a are parallel vectors, find the value of a.   
Since the vectors are parallel, we know that there exists a scalar k such that =k. We can see that k= 4 by dividing the  components of the vector. Therefore, a= -3\*4= -12 and = 8-12 or <8, -12>.

Determine the projections of vectors onto   
1. = 9+ 18- 19 and = 5+ 6-   
proj= •/IIII2  
•= 172  
IIII2= 56  
proj= <215/14, 258/14, -43/14>  
  
2. = -12- 8+ 2 and = 19+ 8- 20  
•= -332  
IIII2= 825  
proj= <-6309/825, -2656/825, 6640/825>  
3. = -5- 10+ 11 and = -19+ 16- 15  
•= -230  
IIII2= 842  
proj= <2185/421, -1840/421, 1725/421>

For the same vectors above, determine the projection of vectors onto   
1. proj= •/IIII2•= 172  
IIII2= 766  
proj=< 1548/766, 3096/766, -3268/766>  
2. •= -332  
IIII2= 212  
proj=<3984/212, 2656/212, -664/212>  
3. •= 842  
IIII2= 246  
proj=< -2105/123, -4210/123, 4631/123>